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# Quasiclassical path integral in coherent-state manifolds

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**Abstract.** Quantization of classical dynamical systems with a Poisson structure on homogeneous Kähler manifolds is considered. The quantization follows the method invented by Berezin and represents the unitary transition operator $\exp(-i\tau H)$ as a quasiclassical path integral in the coherent-state basis. In case the coherent-state manifold appears as a (degenerate) rank-one co-adjoint orbit of the symmetry group, an explicit representation of the transition amplitude in terms of classical data can be derived for large values of the highest weight, which corresponds to the quasiclassical approximation. This representation is further shown to perfectly agree, in contrast to some earlier approaches, with the known exact results and may provide non-trivial asymptotics of physical relevance.

## 1. Introduction

A $G$-homogeneous classical phase space can be thought of as a triple $(M, w; G)$, where $M$ stands for an even-dimensional smooth manifold on which a non-degenerate closed 2-form $w$ is defined. Action of a Lie group $G$ on $M$ is assumed transitive (connects any two points of $M$) and symplectic (leaves $w$ invariant). Put another way, $G$ acts as a group of canonical transformations of $(M, w)$. In fact, any homogeneous symplectic manifold that admits a connected semisimple group of isometries $G$ is locally homeomorphic to a certain co-adjoint orbit of $G$. In view of this, an appropriate quantization of co-adjoint orbits of Lie (super)groups seems to provide an adequate basis to treat associated quantum systems. Quantization of co-adjoint orbits maps under certain conditions a classical $G$-elementary system $(M, w; G)$ into a quantum counterpart $(\mathcal{H}(M), U(G))$, where a Hilbert space $\mathcal{H}(M)$ is constructed out of sections of a complex line bundle over $M$ and $U(G)$ stands for a unitary irreducible representation of $G$ in $\mathcal{H}(M)$. This approach is known as the Kirillov–Kostant geometric quantization [1]. In the case where $M$ is a homogeneous Kähler manifold a general theory of quantization was developed by Berezin [2, 3] in terms of the operator-symbol correspondence and explicitly elaborated for compact Lie groups by Bar-Moshe and Marinov [4]. Path-integral quantization on coherent-state manifolds essentially amounts to Berezin's approach, provided the coherent states form an overcomplete basis in $\mathcal{H}(M)$, which is the case whenever $U(G)$ appears as a unitary irreducible square integrable representation of $G$. It then follows that the corresponding coherent-state manifold $M$ can be viewed as a complex *Kähler* phase space whose metric can be obtained from a single real-valued function on $M$, a Kähler potential $F$. It is important that due to Berezin [2] and Onofri [5] the Kähler potential can be written down explicitly in terms of coherent states. As a result, the path integral can be entirely determined by the Kähler potential (to be more accurate by its analytic continuation) and a classical Hamiltonian (covariant symbol of $H$).

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The purpose of the present paper is to derive a closed representation of the coherent-state path-integral propagator for large values of the dominant weight of the representation $(\mathcal{H}(M), U(G))$. A conventional configuration space path integral could hardly be used for this purpose, since certain constraints are to be additionally imposed to fix a representation, which usually results in a severe technical problem. Our consideration holds for a real semisimple Lie group $G$ possessing square integrable representations. As is known, both compact and non-compact groups with discrete series representations fall into this category; the results we obtained are also equally well applied to the widely used Heisenberg–Weyl coherent states parametrized by points of a complex plane $C$.

The plan of the paper is as follows. In section 2, we explain the notation and survey essentials of the coherent-state path-integral representation of a transition amplitude between two generalized coherent states. Section 3 includes some preliminaries and a brief account of earlier results on the quasiclassical evaluation of quantum-mechanical propagators. Section 4 constitutes the main result, a closed quasiclassical formula for the coherent-state propagator. Relations between various quantization schemes are discussed in section 5. A few examples are gathered in section 6 to illustrate the advantages of this method over some earlier approaches. A summary concludes the paper in section 7.

## 2. Coherent-state path integral

As is known, Perelomov's coherent states for a semisimple group $G$ are points of an orbit of a unitary irreducible representation of $G$ in an abstract Hilbert space $\mathcal{H}$ [6]. By choosing an initial state $|0\rangle$ in $\mathcal{H}$, called the fiducial state, the vectors of the corresponding $G$ orbit are parametrized by points of a homogeneous space $M = G/G_0$, where $G_0$ is the isotropy subgroup of $|0\rangle$. In the following we will be interested in the case where $|0\rangle$ appears as a dominant weight vector (highest weight vector up to the Weyl transformation), which corresponds to the quantization in the Kähler (holomorphic) polarization. It then follows that a factor space $G/G_0$ appears as a Kähler manifold, the Kähler potential being directly expressible in terms of the coherent states as follows. Given a coherent state $|z\rangle$ where $z$ belongs to $G/G_0$, we define (locally)

$$F(\bar{z}_1, z_2) = \log \frac{\langle z_1|z_2\rangle}{\langle z_1|0\rangle\langle 0|z_2\rangle} \tag{1}$$

which can be viewed as an analytic continuation of the real-valued function

$$F(\bar{z}, z) = \log |\langle 0|z\rangle|^{-2}. \tag{2}$$

The latter is called the Kähler potential and was introduced in this way by Berezin [2] and Onofri [5]. This function incorporates geometry of the underlying phase space and plays a crucial role in the following.

The phase space $G/G_0$ can be equipped with an invariant supersymplectic 2-form $w$,

$$w \equiv -\mathrm{i}\delta\bar{\delta}F(\bar{z}, z) \tag{3}$$

where $\delta = \mathrm{d}z \otimes \partial/\partial z$ and $\bar{\delta} = \mathrm{d}\bar{z} \otimes \partial/\partial\bar{z}$ such that the exterior derivative $\mathrm{d} \equiv \delta + \bar{\delta}$. A straightforward calculation shows that $w$ is closed, i.e. $\mathrm{d}w = 0$, which means that $G/G_0$ is a symplectic manifold. In other words, it may serve as a classical phase space. In terms of $F$, the metric and $G$-invariant Liouville measure read

$$\mathrm{d}s^2 = g\,\mathrm{d}z\,\mathrm{d}\bar{z} = \partial^2_{\bar{z}z}F(\bar{z}, z)\,\mathrm{d}z\,\mathrm{d}\bar{z} \tag{4}$$

$$\mathrm{d}\mu(\bar{z}, z) = N\partial^2_{\bar{z}z}F(\bar{z}, z)\frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{2\pi\mathrm{i}} \tag{5}$$

where the normalization constant is chosen such that

$$\int |z\rangle\langle z|\,\mathrm{d}\mu = I. \tag{6}$$

Resolution of unity (6) enters into a path integral as a basic ingredient.

Consider the quantum propagator in the $z$-representation

$$\langle z_F|T\exp\left\{-\frac{\mathrm{i}}{\hbar}\int_0^\tau H(s)\,\mathrm{d}s\right\}|z_I\rangle \equiv \mathcal{P}(\bar{z}_F, z_I, ; \tau) \tag{7}$$

which represents Berezin's covariant symbol of the evolution operator, the Hamiltonian $H(s)$ being a polynomial function of the $G$ generators with time-dependent coefficients. In equation (7) $T$ denotes the time-ordering symbol, which is necessary since for different values of $s$ the corresponding Hamiltonians do not commute.

In order to express the transition amplitude by a path integral, we divide the time interval into $N$ small intervals: $\epsilon = \tau/N$ with $N \to \infty$. Let us define

$$s_k = \epsilon k \qquad z_k = z(s_k) \qquad 0 \leqslant k \leqslant N.$$

With the aid of the time discretization together with relation (6) the propagator can be written up to first order in $\epsilon$ in the form

$$\mathcal{P} = \lim_{N\to\infty}\int_{z_0=z_I}^{\bar{z}_N=\bar{z}_F}\prod_{k=1}^{N-1}\mathrm{d}\mu_k\langle z_F|z_{N-1}\rangle\langle z_1|z_I\rangle\prod_{k=2}^{N-1}\langle z_k|z_{k-1}\rangle\exp\left\{-\mathrm{i}\epsilon\sum_{k=1}^{N}H^{\mathrm{cl}}(\bar{z}_k, z_{k-1})\right\} \tag{8}$$

where

$$H^{\mathrm{cl}}(\bar{z}_k, z_{k-1}; s_k) = \frac{\langle z_k|H(s_k)|z_{k-1}\rangle}{\langle z_k|z_{k-1}\rangle} \qquad \mathrm{d}\mu_k = \mathrm{d}\mu(\bar{z}_k, z_k).$$

The variables $z_N$ and $\bar{z}_0$ do not enter path integral (8) and the corresponding Euler–Lagrange equations are accompanied by the boundary conditions $z_0 = z_I$ and $\bar{z}_N = \bar{z}_F$, respectively. The term

$$\langle z_F|z_{N-1}\rangle\langle z_1|z_I\rangle$$

gives rise to the continuum boundary term to be discussed below.

Before proceeding further, a few remarks on representation (8) apply. First, one should note that no 'operator ordering problem' appears here [7, 8]. As is seen from equation (8), the order has explicitly been fixed by the quantization in terms of covariant symbols.

Second, it is important to indicate definitely the class of trajectories the path integral is localized on. For instance, rearranging the kinetic term in (8) enables one to change the class of trajectories that support the path integral [8, 9]. An actual choice is, however, dictated by a specific problem to be solved. For our purposes, it is sufficient to consider path integral (8) to be localized on a space of square integrable paths (some details are given at the end of section 4).

Third, as is known a compact phase space cannot be covered by a single chart. On the other hand, every integral in equation (8) is written in the same local chart. The way out is that the phase space $M$ is $G$-homogeneous (the group $G$ acts on $M$ through biholomorphic isometries), so that a full set of local charts is generated by actions of $G$: any two charts are locally related by $z \to gz$, for some $g \in G$. Since each local chart covers $M$ except for a set of measure zero (with respect to $\mathrm{d}\mu$), one may, by employing an appropriate covering and adjusted $G$-shifts of variables, restrict the integration to the single local chart $\cong C$.

In the continuum limit equation (8) takes on the form

$$\mathcal{P} = \int_{z(0)=z_I}^{\bar{z}(\tau)=\bar{z}_F}\mathrm{D}\mu(z)\exp\Phi. \tag{9}$$

The total action $\Phi$ includes the boundary term $\Gamma$:

$$\Phi = S + \Gamma$$

where

$$S = -\frac{1}{2}\int_0^\tau \left(\dot{z}\frac{\partial F}{\partial z} - \dot{\bar{z}}\frac{\partial F}{\partial \bar{z}}\right)\mathrm{d}s - \frac{\mathrm{i}}{\hbar}\int_0^\tau H^{\mathrm{cl}}(\bar{z}, z)\,\mathrm{d}s \tag{10}$$

$$\Gamma = \frac{1}{2}[F(\bar{z}_F, z(\tau)) + F(\bar{z}(0), z_I) - F(\bar{z}_F, z_F) - F(\bar{z}_I, z_I)]. \tag{11}$$

These equations coincide up to an obvious change in the notation with those of [10].

The continuum representation (11) originates from a specific discontinuity of paths $z(s)$ and $\bar{z}(s)$ at the relevant endpoints. For example, let us introduce $\Delta_k(\epsilon) \equiv z_k - z_{k-1}$. For any trajectory $z(s)$ one has $\lim_{\epsilon\to 0}\Delta_k(\epsilon) = 0$ for all $k$, except that $\lim_{\epsilon\to 0}\Delta_N(\epsilon) \neq 0$, since $z_N = z_F$ where $z_F$ is an arbitrary complex number. Consequently, the corresponding classical trajectory $z_{\mathrm{cl}}(s)$ does not joint the endpoint value $z_F = z(\tau)$. However, instead of explicitly writing out corresponding shifts of the arguments, it is more convenient to consider variables $\bar{z}(s)$ and $z(s)$ to be independent. Formally, this amounts to saying that the initial $|z_I\rangle$ and final $\langle z_F|$ configurations are in different polarizations [11], which necessitates the appearance of the boundary term. For example, consider a classical system specified by the Hamiltonian function $h^{\mathrm{cl}}$ with initial and final configurations being taken in the polarizations generated by $\partial/\partial q$ and $\partial/\partial p$, respectively:

$$\dot{q} = \frac{\partial h^{\mathrm{cl}}(q, p)}{\partial p} \qquad q(\tau) = q_F$$

$$\dot{p} = -\frac{\partial h^{\mathrm{cl}}(q, p)}{\partial q} \qquad p(0) = p_I. \tag{12}$$

These equations follow from the Hamilton principle of stationary action $\delta\phi = 0$, where

$$\phi = \mathrm{i}\int_0^\tau [p\dot{q} - h^{\mathrm{cl}}]\,\mathrm{d}s - \mathrm{i}p_I[q_F - q(0)].$$

Classical equations of motion follow from the Hamilton principle $\delta\Phi = 0$, which yields

$$\dot{\bar{z}} = \mathrm{i}\hbar^{-1}(\partial^2_{\bar{z}z}F)^{-1}\partial_z H^{\mathrm{cl}} \qquad \bar{z}(\tau) = \bar{z}_F$$

$$\dot{z} = -\mathrm{i}\hbar^{-1}(\partial^2_{\bar{z}z}F)^{-1}\partial_{\bar{z}} H^{\mathrm{cl}} \qquad z(0) = z_I. \tag{13}$$

One sees from (13) that the equations of motion are correctly specified by the boundary conditions and define a canonical phase flow associated with $H^{\mathrm{cl}}$.

In view of a rather complicated form of the coherent-state path integral, it would be desirable to obtain simple sufficient criteria for the stationary-phase approximation to be exact. These are provided by the path-integral generalization of the Duistermaat–Heckman (DH) theorem which, loosely speaking, states that the Wentzel–Kramers–Brillouin (WKB) approximation is exact, provided the Hamiltonian flow leaves a metric of the underlying phase space invariant [12], that is

$$\mathcal{L}_{X_H} g = 0 \tag{14}$$

where $\mathcal{L}_{X_H}$ stands for a Lie derivative along a Hamiltonian vector field that generates the flow.

To formally apply the DH theorem, a kinetic term in an action is required to be of the form $\mathrm{i}\int\theta$, where the symplectic 1-form $\theta$ determines $w$ by $\mathrm{d}\theta = w$. This is pursued for the representation (9)–(11) as follows. Let us define

$$\theta = \frac{\mathrm{i}}{2}[\delta F(\bar{z}, z) - \delta F(\bar{z}_F, z) - \bar{\delta}F(\bar{z}, z) + \bar{\delta}F(\bar{z}, z_I)]. \tag{15}$$

By the very construction, $\mathrm{d}\theta = w$. We recall that $\mathrm{d} = \delta + \bar{\delta}$ and $\delta^2 = \bar{\delta}^2 = \delta\bar{\delta} + \bar{\delta}\delta = 0$. A straightforward computation yields

$$S + \Gamma = \mathrm{i}\int \theta - \frac{\mathrm{i}}{\hbar}\int_0^\tau H^{\mathrm{cl}}\,\mathrm{d}s - \frac{1}{2}[F(\bar{z}_F, z_F) + F(\bar{z}_I, z_I) - 2F(\bar{z}_F, z_I)]$$
$$\equiv \mathrm{i}\int \theta - \frac{\mathrm{i}}{\hbar}\int_0^\tau H^{\mathrm{cl}}\,\mathrm{d}s + \log\langle z_F, |z_I\rangle$$

which results in the desired representation,

$$\frac{\langle z_F|T\,\exp[-(\mathrm{i}/\hbar)\int_0^\tau H(s)\,\mathrm{d}s]|z_I\rangle}{\langle z_F|z_I\rangle} = \int_{z(0)=z_I}^{\bar{z}(\tau)=\bar{z}_F} \mathrm{D}\mu(z)\exp\left[\mathrm{i}\int\theta - \frac{\mathrm{i}}{\hbar}\int_0^\tau H^{\mathrm{cl}}\,\mathrm{d}s\right].$$

To avoid possible confusion, we conclude this section with the following remark. The path-integral quantization depicted above provides an example of the so-called quantization-versus-classical-limit procedures. This approach involves a *dequantization* via a classical coherent-state limit as a necessary preliminary step. In other words, we start with the quantum Hamiltonian $H$, evaluate its classical (Poisson) limit through the associated coherent states and then quantize the obtained classical system $(H^{\mathrm{cl}}, G/G_0, w)$ by using the path integral on $G/G_0$, the point being that we start with an *abstract* representation of $H$ and end up with the *explicit* one to be realized in a Hilbert space of the holomorphic sections over the corresponding co-adjoint orbit. To be aware that the coherent states provide us with a true classical limit, one may note that $G$, being semisimple, can be realized by a Poisson action on $M$; in other words, there is a homomorphic map of the Lie algebra of $G$ into the (Poisson) Lie algebra of the corresponding classical observables [11]. Hence, the classical limit exists and, as was shown by Onofri [5], can be evaluated through the coherent-state expectation values.

## 3. Quasiclassical approximation: preliminaries

Quantization of a group action on an orbit implies that the dominant (highest) weight of the corresponding representation now plays the role of the Planck constant $\hbar$, the large values of the dominant weight corresponding to the classical limit. For a spin system with total spin $j$, quasiclassics occurs for large $j$, whereas in the case of $SU(1,1)$ the classical limit corresponds to large values of the occupation numbers [6]. In this regard, a quasiclassical quantization by the coherent-state path integral may provide some non-trivial asymptotics of physical relevance.

Let $G$ be a compact simple Lie group. For any unitary irreducible representation $U^{\boldsymbol{l}}(G)$, its highest weight $\boldsymbol{l}$ is given by a sum of the fundamental weights $\boldsymbol{\omega}^j$ with non-negative integer coefficients

$$\boldsymbol{l} = \sum_{j=1}^r l_j \boldsymbol{\omega}^j$$

where $r$ stands for the rank of a Lie algebra of $G$. It can then be shown that [13]

$$F^{\boldsymbol{l}} = \sum_{j=1}^r l_j F^j \tag{16}$$

where $\{F^j\}$ represent the fundamental Kähler potentials†. The same dependence on $l_j$ holds for the covariant symbols (coherent-state expectation values) of the basic elements of the Lie

† For non-compact groups this holds for discrete representations [14].

algebra of $G$ [4]. In the following, for the sake of simplicity, we will be solely concerned with the case when only a single term in the series contributes to (16), which corresponds to group orbits of rank = 1. Given a group representation $U^{l}(G)$, the number of non-zero nodes $l_j \neq 0$ in the corresponding Dynkin graph may be called the *rank* of $M$ [4]. If all $l_j > 0$ the corresponding orbit is called non-degenerate. To put this another way, the stability algebra (the algebra of $G_0$), being reductive, splits naturally as a direct sum into its centre and semisimple ideal, the dimension of the centre being equal to the rank of an orbit. This is nothing but the number of *white* roots in the respective *painted* Dynkin diagram [15], with *black* roots corresponding to the semisimple part of the stability algebra.

The requirement of rank = 1 by no means restricts one to the groups $G = SU(2)$ and $SU(1,1)$ and corresponding homogeneous spaces $SU(2)/U(1)$ and $SU(1,1)/U(1)$, for there exist one-rank degenerate orbits of higher-rank groups with complex dimensions $\dim_{\mathrm{c}} M > 1$. For instance, consider the $su(5)$ algebra. Painting its Dynkin graph in all possible ways results in the observation that there exist two rank-one $SU(5)$ homogeneous manifolds, namely, $M_1 = SU(5)/U(1) \otimes SU(4)$ and $M_2 = SU(5)/U(1) \otimes SU(2) \otimes SU(3)$ with real dimensions equal to 8 and 12, respectively.

In physics the symplectic form $w$ has the same units as an angular momentum

$$[w] = [\mathrm{kg\ m^2\ s^{-1}}] = [\hbar].$$

Since we assume the coordinates $z$ and $\bar{z}$ to be dimensionless, the form $w$ in equation (3) is implicitly understood to be measured in units of $\hbar$. It is convenient to introduce a new parameter

$$\lambda = \hbar l$$

that represents a physical quantity, whereas $\hbar$ represents the quantum mechanical yard stick with which to measure $\lambda$ [16]. For instance, in the case $(M = S^2, l \in N)$ $\lambda$ represents the intrinsic angular momentum (spin). It is the parameter $\lambda$ that enters into measurable physical quantities, for example energy. In order to keep them fixed in the limit $l \to \infty$, one should simultaneously imply that $\hbar \to 0$†. This explains why the large highest weight $l$ corresponds to a quasiclassical region.

As was already mentioned, the Lie algebra of classical observables may be represented by covariant symbols or momentum maps which are functions on $M$, with the Lie product being the Poisson brackets given by the Kähler structure. In the limit $l \to \infty$ $(\hbar \to 0)$ the algebra of operators (quantal observables) corresponding to the Lie generators of $G$ reduces to the Poisson algebra of functions (momentum maps) on $M$. To put this another way, in the classical limit, orbits in the co-adjoint representations of $G$ emerge, where different representations give rise to different orbits. A method to obtain classical phase spaces (group orbits) for $G = SO(3)$ has been worked out by Lieb [17] and generalized to compact simple groups by Simon [18].

Let $\{L_\alpha\}$ denote a set of generators of $G$. Consider the Hamiltonian

$$H = \sum_{\alpha=1}^{\dim G} \hbar\omega_\alpha^{(1)} L_\alpha + \sum_{\alpha,\beta=1}^{\dim G} \hbar\omega_{\alpha\beta}^{(2)} \phi_{\alpha\beta}^{(2)}(l) L_\alpha L_\beta + \cdots \tag{17}$$

where $\omega_\alpha^{(1)}, \omega_{\alpha\beta}^{(2)}, \ldots$ are some frequencies that may explicitly depend on time and functions $\phi_{\alpha\beta}^{(2)}(l), \ldots$ are chosen to ensure that $H^{\mathrm{cl}}$ linearly depends on $\lambda$. In view of the

† The limit $\hbar \to 0$ means a passage from systems of units well adjusted for describing quantum objects to those which are more suitable for classical objects.

aforementioned property of momentum maps, this automatically holds for the first term†. From equation (16) it then follows that all dependence on $l$ is isolated in a single factor of $(l)$ multiplying the total action $\Phi$, which justifies the application of the stationary phase approximation to the path-integral propagator (9).

To conclude this section, we make a few remarks about already known quasiclassical formulae for quantum-mechanical propagators. The main result in this respect is due to DeWitt and leads to the short-time (and quasiclassical) approximation to a transition amplitude in a Riemannian configuration space [19], which in the flat case goes over to the well known van Vleck–Morette formula. Further references can be found in [20], see also [21].

An inconvenient point is, however, that there is no simple sufficient criteria for this quasiclassical expression to be exact. Since the configuration space path integral in contrast to that for the phase space does not involve any Liouville measure (in a time-lattice discretization), the DH theorem cannot be applied in this case. Yet some 'experimental' observation can be made. As has been noted by Schulman [22] and DeWitt [23], the configuration space path integral is WKB exact if the expression for the *finite* time propagator coincides with that for the *short* time one. As was shown by Dowker [24], the finite time propagator takes on the same form as the short time one for a free point moving in a space diffeomorphic to the group space of a compact simple group, the explicit formulae being, however, given only for the $SU(N)$ group. For further development of these ideas the recent papers by Inomata *et al* [25] and Junker [26] could be referred to. It is interesting to note that the first construction of the Green function for the general compact Lie group was elaborated by Eskin [27].

When using a short time propagator in the iterative procedure for the path integral one should bear in mind that non-minimal geodesics may exist, indicating that the underlying topology is non-trivial, for example, for a point on $S^1$. Consequently, different homotopic sets of classical trajectories must be summed up, otherwise important information about global geometrical properties of the propagators may be lost. In this regard, correct path-integral representations for propagators on compact Lie groups and spheres have been elaborated explicitly by Marinov and Terentyev in a comprehensive paper [20] in agreement with the Laidlaw–DeWitt–Schulman theorem that establishes a path integral in multiply connected spaces [28].

The coherent-state, path-integral formalism turns out to be more convenient since a powerful machinery of canonical transformations can be employed and a hidden supersymmetry of an action can be revealed, which leads to the path-integral generalization of the DH theorem that provides *universal* simple criteria (14) for the quasiclassics to be exact [12]. Moreover, this method incorporates the underlying symmetry of the problem under consideration, which makes it possible to look for an asymptotic behaviour with respect to the representation indices (eigenvalues of the Casimir operators).

It was Klauder [29], as well as Klauder and Daubechies [30], who first suggested the use of a system of type (13) to derive the semiclassical approximation for the coherent-state path integral. An attempt at the direct evaluation of the Heisenberg–Weyl, coherent-state propagator in the quasiclassical domain was made by Weissman [31]. In the important work

† Numerous spin–spin lattice interactions fall into this category. A simple example of the function $\phi^{(2)}$ is provided by

$$H \sim \frac{1}{2j-1}(J_+^2 + J_-^2) \qquad H^{\mathrm{cl}} \sim 2j = l \qquad \phi^{(2)} \sim \frac{1}{2j-1} \qquad j > 1/2$$

where $\boldsymbol{J}$ stands for the $SU(2)$ generators with $\boldsymbol{J}^2 = j(j+1)$.

by Yaffe [32] a general method for finding classical limits in certain quantum theories was developed. This approach is naturally based upon coherent states associated with a symmetry group and is used to explicitly construct a classical phase space, a corresponding co-adjoint orbit. None of the symmetry groups considered in this paper are semisimple, which makes it necessary to distinguish between adjoint and co-adjoint orbits. Cadavid and Nakashima [33] studied the coherent-state path integral for semisimple Lie algebras, coherent states being sections of a holomorphic line bundle over $G/G_0$. The semiclassical approximation of the quantum evolution operator via coherent states associated with quantized closed curves on the $SU(2)$ orbits was obtained by Karasev and Kozlov [34]. This method was further extended to semisimple Lie algebras [35] and general Kähler phase spaces [36].

## 4. Quasiclassical approximation: coherent-state propagator

In this section we present a derivation of the quasiclassical coherent-state propagator $\mathcal{P}^{\mathrm{qc}}$ by applying the stationary-phase approximation to the path integral (9).

We are looking for the representation

$$\mathcal{P} = \mathrm{e}^{l(\cdots)}[(\cdots) + o(1)] \qquad l \to \infty \tag{18}$$

where $(\cdots)$ stands for $l$-independent functions on a phase space. The quasiclassical propagator is then defined by the leading term of (18)

$$\mathcal{P}^{\mathrm{qc}} = \mathrm{e}^{l(\ldots)}(\cdots). \tag{19}$$

We first rewrite (9) to explicitly take into account the normalization:

$$\mathcal{P} = \langle z_F | z_I \rangle \frac{\int \mathrm{D}\mu(z) \exp \Phi}{\int \mathrm{D}\mu(z) \exp \Phi_0} \qquad \Phi_0 \equiv \Phi|_{H=0}. \tag{20}$$

In order to lift the measure weight factor $\partial^2_{\bar{z}z} F$ in an exponential we make use of a trick that consists of the integration over auxiliary anticommuting fields $\bar{\xi}(t)$ and $\xi(t)$ (see, e.g., [12]):

$$\mathcal{P} = \langle z_F | z_I \rangle \frac{\int \mathrm{D}\bar{z}\, \mathrm{D}z\, \mathrm{D}\bar{\xi}\, \mathrm{D}\xi\, \exp[\Phi + \int \bar{\xi}(s)(\partial^2_{\bar{z}z} F)\xi(s)\, \mathrm{d}s]}{\int \mathrm{D}\bar{z}\, \mathrm{D}z\, \mathrm{D}\bar{\xi}\, \mathrm{D}\xi\, \exp[\Phi_0 + \int \bar{\xi}(s)(\partial^2_{\bar{z}z} F)\xi(s)\, \mathrm{d}s]}. \tag{21}$$

The quasiclassical ($l \to \infty$) motion is described by the approximation

$$\Psi \equiv \Phi + \int \bar{\xi}(\partial^2_{\bar{z}z} F)\xi\, \mathrm{d}s = \Psi|_c + \frac{1}{2}\delta^2\Psi|_c + \cdots \simeq \Psi|_c + \frac{1}{2}\delta^2\Psi|_c \qquad \delta\Psi|_c = 0 \tag{22}$$

and

$$\Psi_0 \equiv \Phi_0 + \int \bar{\xi}(\partial^2_{\bar{z}z} F)\xi\, \mathrm{d}s = \Psi_0|_c + \frac{1}{2}\delta^2\Psi_0|_c \qquad \delta\Psi_0|_c = 0 \tag{23}$$

with the boundary conditions $z(0) = z_I$ and $\bar{z}(\tau) = \bar{z}_F$. The subscript '$c$' denotes a value along the extremals (13).

To proceed further, we introduce variations

$$\delta z \equiv \eta = z - z_c \qquad \delta\bar{z} \equiv \bar{\eta} = \bar{z} - \bar{z}_c$$

which satisfy

$$\eta(0) = 0 \qquad \bar{\eta}(\tau) = 0.$$

It is clear that $\bar{\xi}|_c = \xi|_c = 0$ and in view of (1) $\exp \Phi_0|_c = \langle z_F | z_I \rangle$. Bearing this in mind we insert expansions (22) and (23) into equation (21), perform integrals over $\delta\bar{\xi}$ and $\delta\xi$ coming from $\delta^2\Psi$ and $\delta^2\Psi_0$ that cancel each other† and finally arrive at

$$\mathcal{P}^{\mathrm{qc}}(\bar{z}_F, z_I; \tau) = \mathcal{P}_{\mathrm{red}} \exp \Phi_c \tag{24}$$

where the reduced propagator is given by

$$\begin{aligned} \mathcal{P}_{\mathrm{red}} &= \int \mathrm{D}\eta\, \mathrm{D}\bar{\eta} \exp\left\{ \frac{1}{2} \int_0^\tau (\dot{\bar{\eta}}\eta - \dot{\eta}\bar{\eta})\, \mathrm{d}s - \frac{\mathrm{i}}{2} \int_0^\tau (\eta^2 A + \bar{\eta}^2 C + 2\bar{\eta}\eta B)\, \mathrm{d}s \right\} \\ &= \left( \frac{\mathrm{Det}\, K}{\mathrm{Det}\, K_0} \right)^{-1/2} \end{aligned} \tag{25}$$

with

$$K = \begin{pmatrix} -\mathrm{i}A(s) & -\mathrm{i}B(s) + \partial_s \\ -\mathrm{i}B(s) - \partial_s & -\mathrm{i}C(s) \end{pmatrix} \qquad \text{and} \qquad K_0 = \begin{pmatrix} 0 & \partial_s \\ -\partial_s & 0 \end{pmatrix}.$$

The functions

$$\begin{aligned} A &= \hbar^{-1} \partial_z [(\partial^2_{\bar{z}z} F)^{-1} \partial_z H]|_c \qquad C = \hbar^{-1} \partial_{\bar{z}} [(\partial^2_{\bar{z}z} F)^{-1} \partial_{\bar{z}} H]|_c \\ B &= \frac{1}{2\hbar} \partial_z [(\partial^2_{\bar{z}z} F)^{-1} \partial_{\bar{z}} H]|_c + \frac{1}{2\hbar} \partial_{\bar{z}} [(\partial^2_{\bar{z}z} F)^{-1} \partial_z H]|_c \end{aligned}$$

are calculated with the aid of the Euler–Lagrange equations (13). We recall that $\eta$ and $\bar{\eta}$ are considered to be independent.

Our aim now is to express (25) in terms of the classical orbitals. This can be achieved in a straightforward manner by directly generalizing the derivation of the $SU(2)$ quasiclassical propagator [37] to the case of an arbitrary one-rank Kähler potential. The result reads

$$\mathcal{P}^{\mathrm{qc}}(\bar{z}_F, z_I; \tau) = \exp\left( \Phi_c + \frac{\mathrm{i}}{2} \int_0^\tau B\, \mathrm{d}s \right) \left[ \frac{1}{[g(\bar{z}_c(\tau), z_c(\tau)) g(\bar{z}_c(0), z_c(0))]^{1/2}} \frac{\partial^2 \Phi_c}{\partial \bar{z}_F \partial z_I} \right]^{1/2}. \tag{26}$$

The quasiclassical propagator is thus expressed in terms of the total classical action and classical orbitals and is similar to the DeWitt result for the short time propagator of a particle in a curved configuration space.

There are some important distinctions, however. First, the total action $\Phi$ is involved rather than $S$. The boundary term turns out to be of crucial importance in deriving the correct quasiclassical coherent-state propagator. Were it ignored, the so-called overspecification problem [29] and contradictions with the DH theorem would appear. This was just the case in some earlier attempts to derive the quasiclassical $SU(2)$ propagator [38], whereas the correct expression has recently been obtained [37].

Next, there appears a dependence on the $B$-term. The latter plays the role of normalization and is necessary to fix the quantization (by covariant symbols). This term interpolates between the covariant and contravariant quantization schemes and disappears at the point corresponding to the Weyl quantization. With the aid of the Euler–Lagrange equations it can also be represented in terms of the extremals:

$$B = \frac{\mathrm{i}}{2} (\partial_z \dot{z} - \partial_{\bar{z}} \dot{\bar{z}})|_c = \frac{\mathrm{i}}{2\hbar} (\partial_z X_z - \partial_{\bar{z}} X_{\bar{z}})$$

where $X_H \equiv X_z \partial_z + X_{\bar{z}} \partial_{\bar{z}}$ is the Hamiltonian vector field: $i_{X_H} w + \mathrm{d}H^{\mathrm{cl}} = 0$.

† The non-trivial measure $\mathrm{D}\mu$ contributes to $\mathcal{P}$ at higher orders.

As has already been mentioned, the total action $\Phi$ and the Kähler potential $F$ are proportional to $l$, whereas it is seen that $B \sim l^0$, which agrees with suggestion (19).

Generalization of the above result to the multidimensional case (provided one-rank orbits are still considered!) is straightforward. For instance, consider the compact degenerate $U(N)$ orbit, with complex projective space $CP^{N-1} = SU(N)/SU(N-1) \otimes U(1)$. The complex dimensionality of the manifold is $N-1$ whereas its rank $= 1$. The resulting Kähler potential

$$F = l \log \left[ 1 + \sum_1^{N-1} \bar{z}^i z^i \right]$$

where $\{z^i, i = 1, \ldots, N-1\}$ is a complex vector and a positive integer $l$ is the highest weight specifying a representation. An example is provided by a quantum system with dynamical $U(N)$ symmetry, for example, generated by a set of bilinears $a_i^\dagger a_j, [a_i, a_j^\dagger] = \delta_{ij}, i, j = 1, \ldots, N$, with the quasiclassical parameter being the total number of the field excitations: $l = \sum_i n_i$.

An obvious modification of equation (26) consists of extending (20) and (21) to include vector indices and reads

$$\mathcal{P}^{\mathrm{qc}} = \exp\left(\Phi_c + \frac{\mathrm{i}}{2}\int_0^\tau \mathrm{tr}\, B\, \mathrm{d}s\right)\left[\frac{1}{[g(\bar{z}_c(\tau), z_c(\tau))g(\bar{z}_c(0), z_c(0))]^{1/2}}\, \det\left(\frac{\partial^2 \Phi_c}{\partial \bar{z}_F^i \partial z_I^j}\right)\right]^{1/2} \tag{27}$$

where $g(\bar{z}, z) = \det \partial^2_{\bar{z}^i z^j} F$. As for a possible extension of (27) to higher-rank manifolds (say, maximal orbits with all $l_j \neq 0$), one may note that a corresponding path-integral representation, being a straightforward generalization of (9), is available, for example, for a partition function [9] as well as for a transition amplitude [10]. In point of fact, however, the subsequent application of the stationary-phase approximation necessarily implies that $\|\boldsymbol{l}\|$ must tend to infinity. Technically, different $l_j$ may run to infinity at different rates, including the case when all the components except for a single one are kept constant. Consequently, the straightforward generalization of the above approach to a general case seems to pose a problem.

We conclude this section by the following remark. It may seem that the final result (26) is crucially based on the fact of whether a path integral (9) exists as a bonafide integral. The common belief is, however, that (9) cannot be in general justified as an integral with respect to a certain measure. Moreover, even the justification of the existence of the limit in (8) is rather a non-trivial problem [8], although in the semiclassical domain ($l \to \infty$) this limit under certain restrictions does exist, provided expansion in powers of $1/l$ is first carried out [32]. However, this procedure does not in general lead to a genuine integral with respect to *a path* measure.

Frequently, the statement occurs (see, e.g., [39]) that once the continuum expression (9) is concerned, only formal calculations are possible. This in turn implies that the continuum form of the path integral may at most provide some hint about an actual answer which nevertheless is to be obtained in a rigorous manner within the time-lattice approximation. In general, this assertion is true, although some exceptions, for example, the quasiclassical approximation considered above, are possible. The justification of the continuum representation in the quasiclassical domain ($l \gg 1$) may be given in two formally distinct but in essence similar ways. One may observe, for instance, that a scalar product $\langle z_k | z_{k-1}\rangle$ entering into equation (8) is highly peaked about $\Delta_k(\epsilon) \equiv z_k - z_{k-1} \sim 0$ as $l$ tends to infinity. This implies that, once the leading term $\mathcal{P}^{\mathrm{qc}}$ is concerned, only terms linear in

$\Delta_k$ are to be left in (8), which under some additional mild restrictions on the Hamiltonian function would eventually lead to the continuum representation (25), corrections to the leading term coming, in particular, from higher powers of $\Delta_k$ [32].

We prefer, however, to start from the formal continuum representation (9) which makes sense for continuous differentiable paths. Path integral (9) is localized on a Hilbert space of the square integrable paths, $(z, z) < \infty$ [8], so that in general (and actually, almost surely) $z(s)$ has no time derivative. To find a way out, we represent an arbitrary path in the form $z = z_{\mathrm{cl}}(s) + \delta z(s)$ and expand $\delta z(s)$ in a series over an appropriate basis in $[0, \tau]$. The trick due to Berezin consists of retaining only a finite part of the series, thereby dealing with continuous and differentiable paths at all intermediate steps, with the infinite limit being taken only at the final stage. This procedure converges for the Gaussian path integral (25) [8], thereby justifying the above continuous calculus. This is, however, the case, provided we are concerned as before with a calculation of the leading term, $\mathcal{P}^{\mathrm{qc}}$.

## 5. Other quantization schemes

So far we have been concerned with a specific quantization scheme, the quantization by covariant symbols. The next important quantization scheme to be mentioned here is contravariant quantization. The covariant symbol $H^{\mathrm{cov}}$ which we identify with $H^{\mathrm{cl}}$ is related to the contravariant one $H^{\mathrm{ctr}}$ by [2]

$$H^{\mathrm{cov}}(\bar{z}, z) = \int \exp\{\phi(\bar{z}, z|\bar{v}, v)\} H^{\mathrm{ctr}}(\bar{v}, v)\, \mathrm{d}\mu(\bar{v}, v) \equiv (\hat{T} H^{\mathrm{ctr}})(\bar{z}, z) \tag{28}$$

where

$$\phi(\bar{z}, z|\bar{v}, v) = F(\bar{v}, v) + F(\bar{z}, v) - F(\bar{z}, z) - F(\bar{v}, v). \tag{29}$$

Note that from the resolution of unity (6) it follows that

$$\int \mathrm{e}^{-F(\bar{v}, v)}\, \mathrm{d}\mu(\bar{v}, v) = 1. \tag{30}$$

If the point $(\bar{z}, z)$ is fixed, the potentials $-\phi(\bar{z}, z|\bar{v}, v)$ and $F(\bar{v}, v)$ generate the very same metric and are to be related by a group transformation $g_z$:

$$F(\overline{g_z v}, g_z v) = -\phi(\bar{z}, z|\bar{v}, v). \tag{31}$$

Invariance of the measure $\mathrm{d}\mu$ upon $g_z$ along with the normalization (30) results in

$$1^{\mathrm{cov}} = 1^{\mathrm{ctr}}$$

as it should be.

In principle, the operator $\hat{T}$, being permutable with a group action, can be expressed via the corresponding Casimir operators. In the case under consideration only the second Casimir operator

$$K_2 \equiv \Delta = (\partial^2_{z\bar{z}} F)^{-1} \partial^2_{z\bar{z}} \tag{32}$$

the Laplace–Beltrami operator with respect to the metric (4), is involved.

As is known, in the flat case

$$\hat{T}(\Delta) = \mathrm{e}^{\Delta} \tag{33}$$

whereas for the quantization on a sphere and Lobachevsky plane $\hat{T}(\Delta)$ has been evaluated in a form of infinite products [3].

As is seen from equation (26), both the expression in the brackets and the $B$-term are of the order of $\mathcal{O}(1)$, whereas $\Phi_c = \mathcal{O}(l)$. This means that $H^{\mathrm{ctr}}$ is to be taken with accuracy up to the order of $\mathcal{O}(1)$. Therefore, for our purposes we need an asymptotic relation between symbols rather than the exact one, which is not easily available. As a consequence one may note the following. Being non-positive, the function $\phi$ reaches zero at the point $(\bar{v}, v) = (\bar{z}, z)$. As $l$ goes to infinity, the maximum becomes sharper, localizing $\phi$ at $(\bar{v}, v) = (\bar{z}, z)$. Then expanding the integrand in (28) in powers of $\eta = v - z$ and $\bar{\eta} = \bar{v} - \bar{z}$, one gets

$$\begin{aligned} H^{\mathrm{cov}}(\bar{z}, z) &= \int \mathrm{e}^{-\bar{\eta}\eta}[H^{\mathrm{ctr}}(\bar{z}, z) + \Delta H^{\mathrm{ctr}}(\bar{z}, z)\bar{\eta}\eta]\frac{\mathrm{d}\bar{\eta}\,\mathrm{d}\eta}{2\pi\mathrm{i}} + o(1) \\ &= [1 + \Delta + \mathcal{O}(1/l^2)]H^{\mathrm{ctr}}(\bar{z}, z) \qquad l \to \infty. \end{aligned} \tag{34}$$

In view of this, one may convert equation (26) into a form suitable for the quantization by contravariant symbols.

To conclude this section, we will specify equation (26) for the flat $(M = C)$ case relevant for the Heisenberg–Weyl coherent states. To avoid confusion with dimensions, we introduce, following [16], coordinates $x = q/\alpha$ and $y = p/\beta$ and the complex dimensionless coordinate $z = 1/\sqrt{2}(x + \mathrm{i}y)$. Constants $\alpha$ and $\beta$ are of dimensions of position and momentum, respectively. It is convenient to introduce the dimensionless constant

$$\gamma = \alpha\beta/\hbar$$

which plays the role of the representation index $l$†. The classical limit becomes quite transparent in this notation. It means a passage from systems of units to measure $\alpha$ and $\beta$, which are quite adequate for a quantum description, to those that are more convenient for the classical one. For instance, if one chooses a 'classical scale' $\alpha = 1$ m, $\beta = 1$ kg m s$^{-1}$, then $\gamma^{-1} \approx 10^{-34}$, which effectively corresponds to small $\hbar$. It is just in this sense that one should understand the limit $\hbar \to 0$.

The conventional 2-form $w = \mathrm{d}p \wedge \mathrm{d}q$ goes over to

$$w = -\alpha\beta\,\mathrm{d}x \wedge \mathrm{d}y$$

so that

$$w/\hbar = \mathrm{i}\gamma\,\mathrm{d}\bar{z} \wedge \mathrm{d}z.$$

We introduce a set of $\gamma$-dependent Heisenberg–Weyl coherent states:

$$|z; \gamma\rangle = \exp\left(-\frac{\gamma}{2}\bar{z}z + \sqrt{\gamma}za^{\dagger}\right)|0\rangle \tag{35}$$

hence

$$F = \log|\langle 0|z; \gamma\rangle|^{-2} = \gamma\bar{z}z. \tag{36}$$

In the flat case covariant symbols are related to those in the '$\alpha$-quantization' scheme by

$$H^{\mathrm{cov}}(\bar{z}, z) = (\hat{T}_{\alpha}(\Delta)H^{(\alpha)})(\bar{z}, z) \qquad \hat{T}_{\alpha}(\Delta) = \mathrm{e}^{\alpha\Delta} \qquad \alpha \in [0, 1]$$

the covariant, contravariant and Weyl quantization schemes being specified by $\alpha = 0, 1$ and $1/2$, respectively. As a result one gets

$$H^{\mathrm{cov}} - \tfrac{1}{2}\Delta H^{\mathrm{cov}} \equiv H^{\mathrm{cov}} - [(\tfrac{1}{2} - \alpha) + \alpha]\Delta H^{\mathrm{cov}} = H^{(\alpha)} - (\tfrac{1}{2} - \alpha)\Delta H^{\mathrm{cov}} + \mathcal{O}(1/l^2)$$

† In fact, $\gamma \neq 0$ can be thought of as a number that specifies co-adjoint orbits of the Heisenberg–Weyl group which are two-dimensional planes.

which yields

$$\mathcal{P}^{(\mathrm{qc})}_{\mathrm{flat}} = \left[\frac{1}{\gamma}\frac{\partial^2\Phi_c}{\partial\bar{z}_F\partial z_I}\right]^{1/2} \exp\left\{\Phi_c + \frac{\mathrm{i}}{2}\int_0^\tau B\,\mathrm{d}s\right\}$$

$$= \left[\frac{1}{\gamma}\frac{\partial^2\Phi_c}{\partial\bar{z}_F\partial z_I}\right]^{1/2} \exp\left\{\Phi_c^{(\alpha)} + \mathrm{i}\left(\frac{1}{2}-\alpha\right)\int_0^\tau B\,\mathrm{d}s\right\} \tag{37}$$

$$\sim \left[\frac{1}{\gamma}\frac{\partial^2\Phi_c^{(\alpha)}}{\partial\bar{z}_F\partial z_I}\right]^{1/2} \exp\left\{\Phi_c^{(\alpha)} + \mathrm{i}\left(\frac{1}{2}-\alpha\right)\int_0^\tau B^{(\alpha)}\,\mathrm{d}s\right\} \tag{38}$$

where equivalence classes are defined by $f \sim g = \{f|f/g = 1 + o(1), l \to \infty\}$, so that $H^{(\alpha)} \sim H^{\mathrm{cov}} \equiv H^{\mathrm{cl}}$. All quantities in the above that carry the $\alpha$ index (e.g., $B^{(\alpha)}$) are to be calculated by the substitution $H \to H^{(\alpha)}$.

This result (the first line in equation (37)), with the $B$-term, however, being missed, was derived by Weissman [40] by extending Miller's semiclassical algebra to the coherent-state setting†. Originally, Miller's formalism incorporated eigenstates of Hermitian operators to relate a quantum mechanical matrix element of a general unitary transformation, in the semiclassical limit, to a generator of a corresponding canonical transformation [41].

It is to be noted that in deriving equations (37) and (38) the original equations of motion (13) that correspond to the covariant quantization have been kept fixed. That is why (38) cannot be regarded as a genuine $\alpha$-representation. To derive the latter, one would have to start with equation $\delta\Phi^{(\alpha\neq 0)} = 0$, whose solutions in contrast to (13) would bear an explicit $l$-dependence, namely, $z^{(\alpha)} = z^{\mathrm{cl}} + \mathcal{O}(1/l)$. In that case, however, it would be natural to start with, instead of equation (7), the $\alpha$-symbol of the evolution operator.

## 6. Test examples

In this section, the continuum representation (9) in the semiclassical approximation (26) is shown to directly recover, in contrast to some earlier attempts (see, e.g., [29, 38]), the known exact results, which seems to be a necessary requirement to be met. It is the fact that earlier approaches had certain inconsistencies and failed to recover the exact answers which resulted eventually in the conclusion that the coherent-state path integral does not yield the correct semiclassical results due to intrinsic inevitable ambiguities [42] and, hence, the WKB approximation is not allowed in the case of a matrix element under the coherent-state representation [43]. As we will see shortly, the above statements seem to be somewhat hasty.

### *6.1. Path integral for the Heisenberg–Weyl coherent states*

In view of equation (36), the general representation (9) reduces to [8, 44]

$$\mathcal{P}_\gamma = \langle z_F, \gamma|\exp -\mathrm{i}\int_0^\tau H\,\mathrm{d}s|z_I, \gamma\rangle$$

$$= \int_{z(0)=z_I}^{\bar{z}(\tau)=\bar{z}_F} \mathrm{D}z\,\mathrm{D}\bar{z}\exp\left\{\frac{\gamma}{2}\int_0^\tau (z\dot{\bar{z}} - \bar{z}\dot{z})\,\mathrm{d}s - \mathrm{i}\int_0^\tau H^{\mathrm{cl}}(\bar{z}, z)\,\mathrm{d}s\right.$$

$$\left. + \frac{\gamma}{2}[\bar{z}_F z(\tau) + \bar{z}(0)z_I - |z_F|^2 - |z_I|^2]\right\} \tag{39}$$

with the normalization $\mathcal{P}_\gamma(\bar{z}_F, z_I; \tau)|_{H=0} = \langle z_F, \gamma|z_I, \gamma\rangle$.

† In a subsequent paper [31] the author, to recover the correct result, was forced to take into account the $B$-term in a specific case of the parametric amplifier.

For a harmonic oscillator ($\hbar = 1$) $H = \omega a^\dagger a$, one obtains $H^{\mathrm{cl}} = H^{(\alpha=0)} = \gamma\omega|z|^2$ and $H^{(\alpha)} = H^{\mathrm{cl}} - \alpha\omega$, and solutions to (13) read

$$z_c(s) = z_I \exp(-\mathrm{i}\omega s) \qquad \bar{z}_c(s) = \bar{z}_F \exp(-\mathrm{i}\omega(\tau - s)) \tag{40}$$

which in turn results in

$$\Phi_c^{(\alpha)} = \gamma \bar{z}_F z_I \exp(-\mathrm{i}\omega\tau) - \frac{\gamma}{2}(|z_F|^2 + |z_I|^2) + \mathrm{i}\tau\alpha\omega \qquad B = \omega.$$

Equation (37) can be applied to yield

$$\mathcal{P}_\gamma = \exp \Phi_c^{(\alpha=0)}$$

as it should be.

It also follows from the resolution of unity in terms of (35), that

$$\operatorname{tr} \mathrm{e}^{-\mathrm{i}H\tau} = \gamma \int \frac{\mathrm{d}\bar{z}\,\mathrm{d}z}{2\pi\mathrm{i}} \mathcal{P}(\bar{z}, z) = \gamma \int_0^\infty \mathrm{d}x \exp[-\gamma x(1 - \mathrm{e}^{-\mathrm{i}\omega\tau})] = \frac{\mathrm{e}^{\mathrm{i}\tau\omega/2}}{2\mathrm{i}\sin(\tau\omega/2)}$$

which is a correct answer (cf [43]).

For Gaussian actions the path integral (39) reduces to equation (37). However, in the case when the Hamiltonian cannot be cast into a linear combination of the oscillator group generators $a^\dagger a$, $a^\dagger$ and $a$, the quasiclassical propagator (37) does not merely reduce to the simple form

$$\exp \Phi_c.$$

For instance, for the Hamiltonian of a parametric amplifier

$$H = \omega a^\dagger a - \frac{g}{2}[a^{\dagger 2}\,\mathrm{e}^{-2\mathrm{i}\omega t} + a^2\,\mathrm{e}^{2\mathrm{i}\omega t}]$$

one gets ($\gamma = 1$)

$$z_c(s)\exp(\mathrm{i}\omega s) = \frac{\mathrm{i}\bar{z}_F - z_I \sinh g\tau}{\cosh g\tau} \sinh gs + z_I \cosh gs$$

$$\bar{z}_c(s)\exp(\mathrm{i}\omega(\tau - s)) = \frac{\bar{z}_F + \mathrm{i}z_I \sinh g\tau}{\cosh g\tau} \cosh gs - \mathrm{i}z_I \sinh gs$$

$$\Phi_c = \frac{\bar{z}_F z_I\,\mathrm{e}^{-\mathrm{i}\omega\tau}}{\cosh g\tau} + \frac{\mathrm{i}}{2}\tanh g\tau(\bar{z}_F^2 + z_I^2) - \frac{1}{2}(|z_F|^2 + |z_I|^2) \qquad B = \omega.$$

Equation (37) is again exact and reads

$$\mathcal{P}(\bar{z}_F, z_I; \tau) = (\cosh g\tau)^{-1/2} \exp \Phi_c$$

which coincides with the direct time-lattice calculations [45].

### 6.2. SU(2) path integral

The coherent state for the unitary irreducible representation of the $SU(2)$ group is given by

$$|z; j\rangle = (1 + |z|^2)^{-j} \exp(zJ_+)|j; -j\rangle \tag{41}$$

where $z \in SU(2)/S(U_1 \times U_1) \simeq CP^1$, which can be thought of as an extended complex plane $M = CP^1 = \bar{C}^1$. The operators $J_\pm$ and $J_0$ span the $SU(2)$ algebra

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_0$$

and the lowest weight state $|j; -j\rangle$ is annihilated by $J_-$. From equation (2) it follows that

$$F(\bar{z}, z) = 2j \log(1 + \bar{z}z) \qquad l = 2j \in N$$

where $j$ must be a half-integer corresponding to the unitary irreducible representations of $SU(2)$. From the geometric viewpoint, this requirement is to be imposed in order that a holomorphic prequantum line bundle over $CP^1$ can be constructed. The general equation (9) reads ($\hbar = 1$)

$$\mathcal{P}_j(\bar{z}_F, z_I; \tau) = \int_{z(0)=z_I}^{\bar{z}(\tau)=\bar{z}_F} \mathrm{D}\mu(z) \frac{(1+\bar{z}_F z(\tau))^j (1+\bar{z}(0) z_I)^j}{(1+|z_F|^2)^j (1+|z_I|^2)^j}$$
$$\times \exp\left( j \int_0^\tau \frac{\dot{\bar{z}}(s) z(s) - \bar{z}(s)\dot{z}(s)}{1+\bar{z}(s)z(s)} \,\mathrm{d}s - \mathrm{i}\int_0^\tau H^{\mathrm{cl}}(\bar{z}(s), z(s))\,\mathrm{d}s \right) \tag{42}$$

with the normalization $\mathcal{P}_j|_{H=0} = \langle z_F; j | z_I; j\rangle$. Here $\mathrm{D}\mu_j(z)$ stands for the infinite pointwise product of the $SU(2)$ invariant measures

$$\mathrm{d}\mu_j = \frac{2j+1}{2\pi\mathrm{i}} \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{(1+|z|^2)^2}.$$

As a simple example that directly demonstrates the usefulness of equation (26) consider a system governed by the Hamiltonian [37]

$$H = 2A(t)J_z + f(t)J_+ + \bar{f}(t)J_-.$$

The stationary-phase equations read

$$\mathrm{i}\dot{z} = 2A(t)z + f(t) - \bar{f}(t)z^2 \qquad z(0) = z_I \tag{43}$$
$$-\mathrm{i}\dot{\bar{z}} = 2A(t)\bar{z} + \bar{f}(t) - f(t)\bar{z}^2 \qquad \bar{z}(\tau) = \bar{z}_F. \tag{44}$$

Being of Riccati type, these equations cannot be solved explicitly, but yet some information is available. Solutions to equations (43) and (44) represent the $SU(2)$ phase flows starting from the endpoints $z_I$ and $\bar{z}_F$. This enables one to determine explicitly their dependence on the initial data, which in turn furnishes the necessary information for (26) to be applied [37]:

$$\Phi_c = 2j\log[\bar{a}(\tau) - \bar{b}(\tau)z_I + b(\tau)\bar{z}_F + a(\tau)\bar{z}_F z_I] - j\log(1+|z_F|^2)(1+|z_I|^2)$$
$$B = (2A - f\bar{z} - \bar{f}z)|_c = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\log\frac{-\bar{b}(t)z_I + \bar{a}(t)}{-c(t)\bar{z}_F + d(t)}$$
$$\frac{\partial^2\Phi_c}{\partial\bar{z}_F\partial z_I} = \frac{2j}{[\bar{a}(\tau) - \bar{b}(\tau)z_I + b(\tau)\bar{z}_F + a(\tau)\bar{z}_F z_I]^2}$$

where

$$\dot{a} = -\mathrm{i}Aa + \mathrm{i}f\bar{b} \qquad a(0) = 1$$
$$\dot{b} = -\mathrm{i}Ab - \mathrm{i}f\bar{a} \qquad b(0) = 0. \tag{45}$$

Inserting this into (26) yields

$$\mathcal{P}_j(\bar{z}_F, z_I; \tau) = \exp\Phi_c \tag{46}$$

which coincides with direct time-lattice calculations (cf [38]). This result agrees with the DH theorem. Moreover, the dynamical invariance, i.e. the fact that $H$ belongs to the $SU(2)$ algebra, results in

$$\mathcal{P}_{\mathrm{red}} = 1$$

which is of importance in deriving the generalized Bohr–Sommerfeld quantization conditions [37].

With equation (46) at hand, the partition function

$$Z_j = \int \mathrm{d}\mu_j\,\mathcal{P}_j(\bar{z}, z; \tau)$$

can be easily computed to yield

$$Z_{j=1/2} = \bar{a}(\tau) + a(\tau) \qquad Z_{j=1} = \bar{a}(\tau)^2 + a(\tau)^2 + |a(\tau)|^2 - |b(\tau)|^2.$$

Solutions to equations (45) (all parameters being time-independent) read

$$a(t) = \cos\Omega_0 t - \mathrm{i}\cos(2|f|u)\sin\Omega_0 t \qquad b(t) = -\frac{\mathrm{i}f}{|f|}\sin(2|f|u)\sin\Omega_0 t$$

where

$$\Omega_0 = \sqrt{A^2 + |f|^2} \qquad \cos(2|f|u) = \frac{A}{\Omega_0}.$$

In particular,

$$Z_{j=1/2} = 2\cos\Omega_0\tau \qquad Z_{j=1} = 1 + 2\cos(2\Omega_0\tau)$$

which is a correct result. In a similar fashion the discrete series representations of $SU(1,1)$ may be considered, with the Kähler potential now being $F = 2k\log(1-|z|^2)$, $|z| < 1$, $1 \leqslant 2k \in N$.

As a simple though non-elementary example, we finally consider an asymptotic behaviour of the matrix element

$$\mathcal{P} \equiv \langle z|\exp\left(\frac{\lambda}{2j}J_z^2\right)|z\rangle \qquad \lambda \geqslant 0,\ 2j \gg 1. \tag{47}$$

From equation (41) and the $SU(2)$ commutation relations it follows that

$$\begin{aligned}
\mathcal{P} &= (1+|z|^2)^{-2j}\sum_{n=0}^{2j} C_n^{2j}|z|^{2n}\exp\left(\frac{\lambda}{2j}(n-j)^2\right)\\
&= (1+|z|^2)^{-2j}\,\mathrm{e}^{\lambda j/2}\int_{-\infty}^{\infty}\frac{\mathrm{d}t}{\sqrt{\pi}}\,\mathrm{e}^{-t^2}\sum_{n=0}^{2j}C_n^{2j}|z|^{2n}\exp\left(-\lambda n - \sqrt{\frac{2\lambda}{j}}tn\right)\\
&= (1+|z|^2)^{-2j}\,\mathrm{e}^{\lambda j/2}\int_{-\infty}^{\infty}\frac{\mathrm{d}t}{\sqrt{\pi}}\,\mathrm{e}^{-t^2}\left(1+|z|^2\exp\left(-\lambda - \sqrt{\frac{2\lambda}{j}}t\right)\right)^{2j}\\
&\equiv (1+|z|^2)^{-2j}\,\mathrm{e}^{\lambda j/2}\sqrt{\frac{2j}{\pi}}\int_{-\infty}^{\infty}\mathrm{d}t\,\exp(-2jf(t))
\end{aligned} \tag{48}$$

where

$$f(t) = t^2 - \log(1+|z|^2\exp(-\lambda - 2\sqrt{\lambda}t)).$$

The saddle-point expansion can be further applied to equation (48) to yield

$$\mathcal{P} = \exp\left[-\frac{1}{2}\lambda j k_0^2 + \lambda j k_0 + 2j\log\left(\frac{1+|z|^2\,\mathrm{e}^{-\lambda k_0}}{1+|z|^2}\right)\right]\left(\frac{1}{(1+(\lambda/2)(k_0^2-1))^{1/2}} + o(1)\right) \qquad 2j \to \infty \tag{49}$$

with $k_0$ being the saddle point

$$k_0 = \frac{1-|z|^2\,\mathrm{e}^{-\lambda k_0}}{1+|z|^2\,\mathrm{e}^{-\lambda k_0}}. \tag{50}$$

On the other hand, to apply in this case the basic equation (26), we write down for $\mathcal{P}$ the $SU(2)$ path-integral representation (42) with

$$H^{\mathrm{cl}} = \omega\left[\frac{j}{2}\left(\frac{1-|z|^2}{1+|z|^2}\right)^2 + \frac{|z|^2}{(1+|z|^2)^2}\right] \qquad -\mathrm{i}\tau\omega = \lambda \tag{51}$$

and $\bar{z}_F \equiv \bar{z}$, $z_I \equiv z$. The equations of motion then read

$$\begin{aligned} \dot{z} &= \mathrm{i}\omega z \frac{1-|z|^2}{1+|z|^2}(1-1/(2j)) \qquad z(0) = z \\ \dot{\bar{z}} &= -\mathrm{i}\omega \bar{z} \frac{1-|z|^2}{1+|z|^2}(1-1/(2j)) \qquad \bar{z}(\tau) = \bar{z}. \end{aligned} \tag{52}$$

It can be easily checked that $|z|^2$ appears as a constant of motion, which enables one to integrate (52):

$$z(t) = z\,\mathrm{e}^{\mathrm{i}\Omega t} \qquad \bar{z}(t) = \bar{z}\,\mathrm{e}^{\mathrm{i}\Omega(\tau - t)} \tag{53}$$

where $\Omega$ is to be determined by the relation

$$\frac{\Omega}{\omega} \equiv k = \frac{1-|z|^2\,\mathrm{e}^{-\lambda k}}{1+|z|^2\,\mathrm{e}^{-\lambda k}}\left(1 - \frac{1}{2j}\right). \tag{54}$$

Expanding further both sides of equation (54) in the inverse powers of $2j$ yields

$$k = k_0 + k_1/(2j) + \cdots$$

where

$$k_0 = \frac{1-|z|^2\,\mathrm{e}^{-\lambda k_0}}{1+|z|^2\,\mathrm{e}^{-\lambda k_0}} \tag{55}$$

which coincides with equation (50) and

$$k_1 = \left(\frac{\lambda}{2}(1-k_0^2) - 1\right)k_0^{-1}. \tag{56}$$

Now that we have explicitly determined classical solutions, the basic ingredients entering into (26) can easily be obtained. In the first place, with the help of equations (53), (55) and (56) one gets

$$\begin{aligned} \Phi_c &= -\frac{\lambda}{2} j k_0^2 + \lambda j k_0 + 2j \log\left(\frac{1+|z|^2\,\mathrm{e}^{-\lambda k_0}}{1+|z|^2}\right) + \frac{\lambda(1-k_0^2)}{4} + \mathcal{O}(1/j) \\ \frac{\mathrm{i}}{2}\int_0^\tau B\,\mathrm{d}t &= \frac{\lambda k_0}{2} - \frac{\lambda(1-k_0^2)}{4} + \mathcal{O}(1/j). \end{aligned} \tag{57}$$

A few laborious calculations can be carried out to obtain

$$\left.\frac{\partial^2 \Phi_c}{\partial \bar{z}_F \partial z_I}\right|_{\bar{z}_F = \bar{z}, z_I = z}.$$

The point is that when calculating $\partial k_0/\partial \bar{z} \equiv K_{\bar{z}}$, $\partial k_0/\partial z \equiv K_z$ and $\partial^2 k_0/\partial \bar{z}\partial z \equiv K_{\bar{z}z}$ one has to bear in mind that $k_0$ implicitly depends on the endpoints through equation (55). After some algebra one, however, finds that

$$\begin{aligned} K_z &= -\frac{2\bar{z}\,\mathrm{e}^{-\lambda k_0}}{1+|z|^2\,\mathrm{e}^{-\lambda k_0}} \qquad K_{\bar{z}} = -\frac{2z\,\mathrm{e}^{-\lambda k_0}}{1+|z|^2\,\mathrm{e}^{-\lambda k_0}} \\ K_{\bar{z}z} &= -\frac{2k_0}{(1-a)^3}\frac{\mathrm{e}^{-\lambda k_0}}{(1+|z|^2\,\mathrm{e}^{-\lambda k_0})^2} \qquad a \equiv \frac{\lambda}{2}(1-k_0^2). \end{aligned}$$

The first line of equation (57) along with the derivatives obtained lead to

$$\begin{aligned} \left.\frac{\partial^2 \Phi_c}{\partial \bar{z}_F \partial z_I}\right|_{\bar{z}_F = \bar{z}, z_I = z} &= -\lambda j K_{\bar{z}z}(k_0 - 1) - \lambda j K_z K_{\bar{z}} - \frac{2j}{(1+k_0)^2}[K_{\bar{z}z}(1+k_0) - K_z K_{\bar{z}}] + \mathcal{O}(1) \\ &= \frac{2j\,\mathrm{e}^{-\lambda k_0}}{(1-a)^2(1+|z|^2 e^{-\lambda k_0})^2}\left[\left(\frac{1}{(1+k_0)^2} - \frac{1}{2}\right)(1-k_0^2)\right. \end{aligned}$$

$$-\left(\frac{\lambda(1-k_0)}{2}-\frac{1}{1+k_0}\right)\frac{2k_0}{1-a}\Bigg]+\mathcal{O}(1)$$
$$=\frac{2j\,\mathrm{e}^{-\lambda k_0}}{(1-a)(1+|z|^2\,\mathrm{e}^{-\lambda k_0})^2}+\mathcal{O}(1). \qquad (58)$$

Inserting equations (57) and (58) into equation (26) immediately results in (49).

## 7. Conclusion

In the present paper, we have discussed quasiclassical quantization of classical mechanics on a symplectic group orbit of rank $=1$ in terms of the relevant coherent-state path integral, which provides an appropriate expansion of a quantum-mechanical propagator for large values of the highest weight $l$ that specifies the underlying group representation. The principal result is a new explicit quasiclassical formula for a propagator on a coherent-state manifold, which is written entirely in terms of classical data and reveals the leading large $l$ behaviour of the propagator. This representation is important since a wealth of physically relevant classical phase spaces admit a natural Kähler polarization, for example $S^2 \simeq CP^1$, the classical phase space for a spin, or $S^{1,1} \simeq D^1$, a unit disk on a complex plane—a natural phase space for models of quantum optics [25]. In this regard, the quasiclassical representation (26) can be applied to the study of, for example, spin tunnelling in the semiclassical limit [46] and related problems [47] as well as to the behaviour of highly excited field states in quantum optical models.

As for possible generalizations of (26), it would be of practical importance to modify it to include non-local actions that arise provided certain degrees of freedom in an original Hamiltonian can be integrated out. This is the case for a large class of interactions that involve bilinear combinations of Lie algebra generators and field coordinates. An example is provided by the spin-radiation/atom-radiation interaction:

$$H=\omega J_z+\omega a^\dagger a+\frac{g}{\sqrt{2j}}(J_+a+J_-a^\dagger) \qquad [a,a^\dagger]=1$$

where we have for the sake of notational simplicity put equal the frequencies of the field and spin oscillations. Consider

$$\mathcal{P}\equiv\langle z|\mathrm{tr}_{a,a^\dagger}\,\mathrm{e}^{-\mathrm{i}\tau H}|z\rangle \qquad \mathcal{P}|_{H=0}=1.$$

Having integrated out the field degrees of freedom, one is left with an effective action whose non-local part looks like

$$2jg^2\int_0^\tau\frac{\bar z(t)G(t-s)z(s)}{(1+|z(t)|^2)(1+|z(s)|^2)}\,\mathrm{d}t\,\mathrm{d}s \qquad G(t-s)=\sum_{n=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\omega_n(t-s)}}{\mathrm{i}\omega_n-\mathrm{i}\omega} \qquad \omega_n=\frac{2\pi}{\tau}n.$$

Equation (26) cannot be directly applied in this case unless a few simplifications are carried out. For instance, in the local limit that holds, roughly speaking, as $\omega\to\infty$ one has

$$G(t-s)\to\frac{\mathrm{i}}{\omega}\delta(t-s)$$

so that one arrives at the $SU(2)$ path-integral representation (42) with

$$H^{\mathrm{cl}}=-j\omega\frac{1-|z|^2}{1+|z|^2}-2j(g^2/\omega)\frac{|z|^2}{(1+|z|^2)^2}.$$

The equations of motion then read

$$\dot z=-\mathrm{i}\Omega z \qquad z(0)=z \qquad \dot{\bar z}=\mathrm{i}\Omega\bar z \qquad \bar z(\tau)=\bar z$$

where

$$\frac{\Omega}{\omega} \equiv k = 1 - \alpha^2 \frac{1 - |z|^2\,\mathrm{e}^{-\lambda k}}{1 + |z|^2\,\mathrm{e}^{-\lambda k}} \qquad \alpha^2 = (g/\omega)^2 \qquad \lambda = \mathrm{i}\omega\tau.$$

The solution to the above equations is quite transparent

$$k = 1 - \alpha^2 \frac{1 - |z|^2\,\mathrm{e}^{-\lambda}}{1 + |z|^2\,\mathrm{e}^{-\lambda}} + \mathcal{O}(\alpha^4) \qquad \alpha^2 \to 0.$$

Proceeding further as above one, finally obtains

$$\mathcal{P} = \exp \Phi_c (1 + o(1)) \qquad 2j \gg 1, \alpha^2 \ll 1, 2j\alpha^2 \text{ fixed}$$

where

$$\Phi_c = j\lambda + 2j \log \frac{1 + |z|^2\,\mathrm{e}^{-\lambda}}{1 + |z|^2} + \frac{\lambda 2 j \alpha^2 |z|^2\,\mathrm{e}^{-\lambda}}{(1 + |z|^2\,\mathrm{e}^{-\lambda})^2}.$$

It would also be interesting to extend this approach to the supersetting, since even the simplest super phase spaces happen to be relevant to important physics. For instance, the one-rank degenerate orbit of the $SU(2|1)$ supergroup can be viewed as a phase space of the $t - J$ model of strongly correlated electrons which is believed to adequately describe a high-$T_c$ superconducting state.

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